

# SUBHARMONIC FUNCTIONS, MEAN VALUE INEQUALITY, BOUNDARY BEHAVIOR, NONINTEGRABILITY AND EXCEPTIONAL SETS

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**ABSTRACT.** We begin by shortly recalling a generalized mean value inequality for subharmonic functions, and two applications of it: first a weighted boundary behavior result (with some new references and remarks), and then a borderline case result to Suzuki's nonintegrability results for superharmonic and subharmonic functions. The main part of the talk consists, however, of partial improvements to Blanchet's removable singularity results for subharmonic, plurisubharmonic and convex functions.

## 1. INTRODUCTION

1.1. In section 2 and 3 we give refinements (Theorems 1 and 2) to our previous results concerning generalized mean value inequalities for subharmonic functions and its applications on the boundary behavior. In section 4 we remark that there exists a limiting case result (Theorem 3 and Corollary 3) for Suzuki's results on the nonintegrability of superharmonic and subharmonic functions. The main part of the article is, however, section 5, where we give partial improvements to Blanchet's removable singularity results for subharmonic, plurisubharmonic and convex functions (Theorems 4, 5 and Corollaries 4, 5 and 6).

1.2. **Notation.** Our notation is more or less standard, see [Ri99]. However, for the convenience of the reader we recall here the following. We use the common convention  $0 \cdot \infty = 0$ .  $B(x, r)$  is the Euclidean ball in  $\mathbb{R}^n$  with center  $x$  and radius  $r$ . We write  $v_n = m(B(0, 1))$ , where  $m$  is the Lebesgue measure in  $\mathbb{R}^n$ . In integrals we will write also  $dx$  for the Lebesgue measure. We identify  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$ . Let  $0 \leq \alpha \leq n$  and  $A \subset \mathbb{R}^n$ ,  $n \geq 1$ . Then we write  $\mathcal{H}^\alpha(A)$  for the  $\alpha$ -dimensional Hausdorff (outer) measure of  $A$ . Recall that  $\mathcal{H}^0(A)$  is the number of points of  $A$ . In sections 2, 3 and 4  $\Omega$  is always a domain in  $\mathbb{R}^n$ ,  $\Omega \neq \mathbb{R}^n$ ,  $n \geq 2$ . In section 5  $\Omega$  is either a domain in  $\mathbb{R}^n$  or in  $\mathbb{C}^n$ ,  $n \geq 2$ . The diameter of  $\Omega$  is denoted by  $\text{diam } \Omega$ . The distance from  $x \in \Omega$  to  $\partial\Omega$ , the boundary of  $\Omega$ , is denoted by  $\delta(x)$ .  $\mathcal{L}_{\text{loc}}^p(\Omega)$ ,  $p > 0$ , is the space of functions  $u$  in  $\Omega$  for which  $|u|^p$  is locally integrable on  $\Omega$ . Our constants  $C$  are always positive, mostly  $\geq 1$ , and they may vary from line to line. If  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $n \geq 2$ , and  $j \in \mathbb{N}$ ,  $1 \leq j \leq n$ , then we write  $x = (x_j, X_j)$ , where  $X_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ . Moreover, if  $A \subset \mathbb{R}^n$ ,  $1 \leq j \leq n$ , and  $x_j^0 \in \mathbb{R}$ ,  $X_j^0 \in \mathbb{R}^{n-1}$ , we write

$$A(x_j^0) = \{X_j \in \mathbb{R}^{n-1} : x = (x_j^0, X_j) \in A\}, \quad A(X_j^0) = \{x_j \in \mathbb{R} : x = (x_j, X_j^0) \in A\}.$$

We will use similar notation in  $\mathbb{C}^n$ ,  $n \geq 2$ , when considering separately subharmonic and plurisubharmonic functions.

For the definition and properties of subharmonic, separately subharmonic, plurisubharmonic and convex functions, see e.g. [Ra37], [Le69], [Hel69], [Her71], [Hö94] and [We94].

## 2. THE MEAN VALUE INEQUALITY

2.1. **Previous results.** If  $u$  is a nonnegative and subharmonic function on  $\Omega$ , and  $p > 0$ , then there is a constant  $C = C(n, p) \geq 1$  such that

$$(1) \quad u(x)^p \leq \frac{C}{m(B(x, r))} \int_{B(x, r)} u(y)^p dm(y)$$

for all  $B(x, r) \subset \Omega$ .

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See [FeSt72, Lemma 2, p. 172], [Ku74, Theorem 1, p. 529], [Ga81, Lemma 3.7, pp. 121-123], [AhRu93, (1.5), p. 210]. These authors considered only the case when  $u = |v|$  and  $v$  is a harmonic function. However, the proofs in [FeSt72] and [Ga81] apply verbatim also in the general case of nonnegative subharmonic functions. This was pointed out in [Ri89, Lemma, p. 69], [Su90, p. 271], [Su91, p. 113], [Ha92, Lemma 1, p. 113], [Pa94, p. 18] and [St98, Lemma 3, p. 305]. In [AhBr88, p. 132] it was pointed out that a modification of the proof in [FeSt72] gives in fact a slightly more general result, see 2.2 below. A possibility for an essentially different proof was pointed out already in [To86, pp. 188-190]. Later other different proofs were given in [Pa94, p. 18, and Theorem 1, p. 19] (see also [Pa96, Theorem A, p. 15]), [Ri99, Lemma 2.1, p. 233], [Ri00] and [Ri01, Theorem, p. 188]. The results in [Pa94], [Ri99], [Ri00] and [Ri01] hold in fact for more general function classes than just for nonnegative subharmonic functions. See 2.2 and Corollary 1 below. Compare also [DBTr84] and [Do88, p. 485].

The inequality (1) has many applications. Among others, it has been applied to the (weighted) boundary behavior of nonnegative subharmonic functions [To86, p. 191], [Ha92, Theorems 1 and 2, pp. 117-118], [St98, Theorems 1, 2 and 3, pp. 301, 307], [Ri99, Theorem, p. 233], [Ri00], and on the nonintegrability of subharmonic and superharmonic functions [Su90, Theorem 2, p. 271], [Su91, Theorem, p. 113].

Because of the importance of (1), it is worthwhile to present a unified result which contains this mean value inequality and all its above referred generalizations. Such a generalization is proposed below in Theorem 1. In order to state our result and unify the terminology, we give first two definitions.

**2.2. Quasi-nearly subharmonic functions.** We call a (Lebesgue) measurable function  $u : \Omega \rightarrow [-\infty, \infty]$  *quasi-nearly subharmonic*, if  $u \in \mathcal{L}_{\text{loc}}^1(\Omega)$  and if there is a constant  $C_0 = C_0(n, u, \Omega) \geq 1$  such that

$$(2) \quad u(x) \leq \frac{C_0}{r^n} \int_{B(x, r)} u(y) dm(y)$$

for any ball  $B(x, r) \subset \Omega$ . Compare [Ri99, p. 233] and [Do57, p. 430]. Nonnegative quasi-nearly subharmonic functions have previously been considered in [Pa94] (Pavlović called them "functions satisfying the  $sh_K$ -condition") and in [Ri99], [Ri00] (where they were called "pseudosubharmonic functions"). See [Do88, p. 485] for an even more general function class of (nonnegative) functions. As a matter of fact, also we will restrict ourselves to nonnegative functions.

Nearly subharmonic functions, thus also quasisubharmonic and subharmonic functions, are examples of quasi-nearly subharmonic functions. Recall that a function  $u \in \mathcal{L}_{\text{loc}}^1(\Omega)$  is *nearly subharmonic*, if  $u$  satisfies (2) with  $C_0 = \frac{1}{v_n}$ , see [Her71, pp. 14, 26]. Furthermore, if  $u \geq 0$  is subharmonic and  $p > 0$ , then by (1) above,  $u^p$  is quasi-nearly subharmonic. By [Pa94, Theorem 1, p. 19] or [Ri99, Lemma 2.1, p. 233] this holds even if  $u \geq 0$  is quasi-nearly subharmonic. See also [AhBr88, p. 132].

**2.3. Permissible functions.** A function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is called *permissible*, if there is a nondecreasing, convex function  $\psi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and an increasing surjection  $\psi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\psi = \psi_2 \circ \psi_1$  and such that the following conditions are satisfied:

- (a)  $\psi_1$  satisfies the  $\Delta_2$ -condition.
- (b)  $\psi_2^{-1}$  satisfies the  $\Delta_2$ -condition.
- (c) The function  $t \mapsto \frac{t}{\psi_2(t)}$  is *quasi-increasing*, i.e. there is a constant  $C = C(\psi_2) \geq 1$  such that

$$\frac{s}{\psi_2(s)} \leq C \frac{t}{\psi_2(t)}$$

for all  $s, t \in \mathbb{R}_+$ ,  $0 \leq s \leq t$ .

Observe that the condition (b) is equivalent with the following condition.

- (b') For some constant  $C = C(\psi_2) \geq 1$ ,

$$\psi_2(Ct) \geq 2\psi_2(t)$$

for all  $t \in \mathbb{R}_+$ .

Recall that a function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies the  $\Delta_2$ -condition, if there is a constant  $C = C(\psi) \geq 1$  such that

$$\psi(2t) \leq C\psi(t)$$

for all  $t \in \mathbb{R}_+$ .

If  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an increasing surjection satisfying the conditions (b) and (c), we say that it is *strictly permissible*. Permissible functions are necessarily continuous.

Let it be noted that the condition (c) above is indeed natural. For just one counterpart to it, see e.g. [HiPh57, Theorem 7.2.4, p. 239].

Observe that our previous definition for permissible functions in [Ri99, 1.3, p. 232] was *much more restrictive*: A function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  was there defined to be *permissible* if it is of the form  $\psi(t) = \vartheta(t)^p$ ,  $p > 0$ , where  $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing, convex function satisfying the  $\Delta_2$ -condition.

**2.4. Examples of permissible functions.** The simple example below in (vi), shows that functions of type (ii) are by no means the only permissible functions. The variety of permissible functions is of course wide.

- (i) The functions  $\psi_1(t) = \vartheta(t)^p$ ,  $p > 0$ .
- (ii) Functions of the form  $\psi_2 = \phi_2 \circ \phi_2$ , where  $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a concave surjective function whose inverse  $\phi_2^{-1}$  satisfies the  $\Delta_2$ -condition, and  $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function satisfying the  $\Delta_2$ -condition. (Observe here that any concave function  $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is necessarily nondecreasing.)
- (iii)  $\psi_3(t) = ct^{p\alpha}[\log(\delta + t^{p\gamma})]^\beta$ , where  $c > 0$ ,  $0 < \alpha < 1$ ,  $\delta \geq 1$ , and  $\beta, \gamma \in \mathbb{R}$  are such that  $0 < \alpha + \beta\gamma < 1$ , and  $p \geq 1$ .
- (iv) For  $0 < \alpha < 1$ ,  $\beta \geq 0$  and  $p \geq 1$ ,

$$\psi_4(t) = \begin{cases} p^\beta t^{p\alpha}, & \text{for } 0 \leq t \leq e, \\ t^{p\alpha}(\log t^p)^\beta, & \text{for } t > e. \end{cases}$$

- (v) For  $0 < \alpha < 1$ ,  $\beta < 0$  and  $p \geq 1$ ,

$$\psi_5(t) = \begin{cases} \left(\frac{-\beta p}{\alpha}\right)^\beta t^{p\alpha}, & \text{for } 0 \leq t \leq e^{-\beta/\alpha}, \\ t^{p\alpha}(\log t^p)^\beta, & \text{for } t > e^{-\beta/\alpha}. \end{cases}$$

- (vi) For  $p \geq 1$ ,

$$\psi_6(t) = \begin{cases} 2n + \sqrt{t^p - 2n}, & \text{for } t^p \in [2n, 2n+1), \quad n = 0, 1, 2, \dots, \\ 2n + 1 + [t^p - (2n+1)]^2, & \text{for } t^p \in [2n+1, 2n+2), \quad n = 0, 1, 2, \dots. \end{cases}$$

For  $p = 1$  the functions in (i), (iii), (iv), (v), (vi), and also in (ii) provided that  $\varphi_2(t) = t$ , are strictly permissible.

Observe that our previous results were restricted to the cases where  $\psi$  was either of type (i) ([Ri99, (1.3), p. 232, and Lemma 2.1, p. 233]) or of type (ii) ([Ri01, Theorem, p. 188]).

**2.5. The generalized mean value inequality.** The result (which was presented also at the NORDAN 2000 Meeting, see [Ri00]) is the following. Its proof is a modification of Pavlović's argument [Pa94, proof of Theorem 1, p. 20].

**Theorem 1.** *Let  $u$  be a nonnegative quasi-nearly subharmonic function on  $\Omega$ . If  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a permissible function, then  $\psi \circ u$  is quasi-nearly subharmonic on  $\Omega$ , i.e. there exists a constant  $C = C(n, \psi, u) \geq 1$  such that*

$$\psi(u(x_0)) \leq \frac{C}{\rho^n} \int_{B(x_0, \rho)} \psi(u(y)) dm(y)$$

for any ball  $B(x_0, \rho) \subset \Omega$ .

*Proof.* In view of [Ri99, Lemma 2.1, p. 233] we may restrict us to the case where  $\psi = \psi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is strictly permissible. Since  $\psi$  is continuous,  $\psi \circ u$  is measurable and  $\psi \circ u \in \mathcal{L}_{loc}^1(\Omega)$ . It remains to show that  $\psi \circ u$  satisfies the generalized mean value inequality (2). But this can be seen exactly as in [Ri01, proof of Theorem, pp. 188-189], the only difference being that instead of the property 2.4 in [Ri01, p. 188] of concave functions, one now uses the above property (c) in 2.3 of permissible functions.  $\square$

**Corollary 1.** ([Ri01, Theorem, p. 188]) *Let  $u$  be a nonnegative subharmonic function on  $\Omega$ . Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a concave surjection whose inverse  $\psi^{-1}$  satisfies the  $\Delta_2$ -condition. Then there exists a constant  $C = C(n, \psi, u) \geq 1$  such that*

$$\psi(u(x_0)) \leq \frac{C}{\rho^n} \int_{B(x_0, \rho)} \psi(u(y)) dm(y)$$

for any ball  $B(x_0, \rho) \subset \Omega$ .

### 3. WEIGHTED BOUNDARY BEHAVIOR

3.1. Before giving our first application of Theorem 1, we recall some terminology from [Ri99, pp. 231–232].

3.2. **Admissible functions.** A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is *admissible*, if it is increasing (strictly), surjective, and there are constants  $C_2 > 1$  and  $r_2 > 0$  such that

$$\varphi(2t) \leq C_2 \varphi(t) \quad \text{and} \quad \varphi^{-1}(2s) \leq C_2 \varphi^{-1}(s) \quad \text{for all } s, t \in \mathbb{R}_+, 0 \leq s, t \leq r_2.$$

Nonnegative, nondecreasing functions  $\varphi_1(t)$  which satisfy the  $\Delta_2$ -condition and for which the functions  $t \mapsto \frac{\varphi_1(t)}{t}$  are nondecreasing, are examples of admissible functions. Further examples are  $\varphi_2(t) = ct^\alpha[\log(\delta + t^\gamma)]^\beta$ , where  $c > 0$ ,  $\alpha > 0$ ,  $\delta \geq 1$ , and  $\beta, \gamma \in \mathbb{R}$  are such that  $\alpha + \beta\gamma > 0$ .

3.3. **Accessible boundary points and approach regions.** Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an admissible function and let  $\alpha > 0$ . We say that  $\zeta \in \partial\Omega$  is  $(\varphi, \alpha)$ -accessible, if

$$(3) \quad \Gamma_\varphi(\zeta, \alpha) \cap B(\zeta, \rho) \neq \emptyset$$

for all  $\rho > 0$ . Here

$$\Gamma_\varphi(\zeta, \alpha) = \{x \in \Omega : \varphi(|x - \zeta|) < \alpha\delta(x)\},$$

and we call it a  $(\varphi, \alpha)$ -approach region in  $\Omega$  at  $\zeta$ .

3.4. **Remarks.** (a) In the case when  $\varphi(t) = t$ , the condition (3) is often called the *corkscrew condition*. See e.g. [JeKe82, p. 93].

(b) It follows from [NäVä91, 2.19, p. 14] that *all* boundary points of *any* John domain are  $(\varphi, \alpha)$ -accessible for some  $\alpha > 0$  (where  $\alpha$  depends of course of the parameters of the John domain), provided the admissible function  $\varphi$  satisfies an additional condition,

$$(4) \quad \sup\left\{\frac{\varphi(t)}{t} : 0 < t < r_2\right\} < \infty.$$

Recall that bounded NTA domains, bounded  $(\varepsilon, \delta)$ -domains of Jones, and more generally uniform domains are John domains, see [NäVä91] and the references therein. Therefore, using different admissible functions one obtains various types of approach, and in certain cases also non-tangential approach, see [St98, pp. 302–304]. Examples of admissible functions satisfying this additional condition (4) are nonnegative, nondecreasing functions  $\varphi_1(t)$  which satisfy the  $\Delta_2$ -condition and for which the functions  $t \mapsto \frac{\varphi_1(t)}{t}$  are nondecreasing (for small arguments). Further examples are  $\varphi_2(t) = ct^\alpha[\log(\delta + t^\gamma)]^\beta$ , where  $c > 0$ ,  $\alpha > 1$ ,  $\delta \geq 1$ , and  $\beta, \gamma \in \mathbb{R}$  are such that  $\alpha - 1 + \beta\gamma > 0$ .

(c) Mizuta [Mi91] has considered boundary limits of harmonic functions in Sobolev-Orlicz classes on bounded Lipschitz domains  $U$  of  $\mathbb{R}^n$ ,  $n \geq 2$ . His approach regions are of the form

$$\Gamma_\phi(\zeta, \alpha) = \{x \in U : \phi(|x - \zeta|) < \alpha\delta(x)\},$$

where now  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing function which satisfies the  $\Delta_2$ -condition and is such that  $t \mapsto \frac{\phi(t)}{t}$  is nondecreasing. As pointed out above, such functions are admissible in our sense, and they satisfy also the above condition (4). In fact, they form a proper subclass of our admissible functions.

3.5. **The weighted boundary behavior result.** Below is the refinement to our previous result [Ri99, Theorem, p. 233]. This result was presented also at the NORDAN 2000 Meeting [Ri00], and it improves the previous results of Gehring [Ge57, Theorem 1, p. 77], Hallenbeck [Ha92, Theorems 1 and 2, pp. 117–118] and Stoll [St98, Theorem 2, p. 307].

**Theorem 2.** *Let  $\mathcal{H}^d(\partial\Omega) < \infty$  where  $0 \leq d \leq n$ . Suppose that  $u$  is a nonnegative quasi-nearly subharmonic function in  $\Omega$ . Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an admissible function and  $\alpha > 0$ . Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a permissible function. Suppose that*

$$\int_{\Omega} \psi(u(x))\delta(x)^\gamma dm(x) < \infty$$

for some  $\gamma \in \mathbb{R}$ . Then

$$\lim_{\rho \rightarrow 0} \left( \sup_{x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} \{ \delta(x)^{n+\gamma} [\varphi^{-1}(\delta(x))]^{-d} \psi(u(x)) \} \right) = 0$$

for  $\mathcal{H}^d$ -almost every  $(\varphi, \alpha)$ -accessible point  $\zeta \in \partial\Omega$ . Here

$$\Gamma_{\varphi, \rho}(\zeta, \alpha) = \{x \in \Gamma_\varphi(\zeta, \alpha) : \delta(x) < \rho\}.$$

The proof is verbatim the same as [Ri99, proof of Theorem, pp. 235–238], except that now we just replace [Ri99, Lemma 2.1, p. 233] by the more general Theorem 1 above.  $\square$

**Remark.** (Added in December 2003) Mizuta has given a similar result (for the case when  $\psi(t) = t^p$ ,  $p > 0$ , and  $\phi(t) = t^q$ ,  $q \geq 1$ ) with a different proof, see [Mi01, Theorem 2, p. 73].

**Corollary 2A.** *Let  $\Omega$  be a John domain and let  $\mathcal{H}^d(\partial\Omega) < \infty$  where  $0 \leq d \leq n$ . Suppose that  $u$  is a nonnegative quasi-nearly subharmonic function in  $\Omega$ . Let  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an admissible function satisfying the additional condition (4) above. Let  $\alpha > 0$ . Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a permissible function. Suppose that*

$$\int_{\Omega} \psi(u(x)) \delta(x)^\gamma dm(x) < \infty$$

for some  $\gamma \in \mathbb{R}$ . Then

$$\lim_{\rho \rightarrow 0} \left( \sup_{x \in \Gamma_{\phi, \rho}(\zeta, \alpha)} \{ \delta(x)^{n+\gamma} [\phi^{-1}(\delta(x))]^{-d} \psi(u(x)) \} \right) = 0$$

for  $\mathcal{H}^d$ -almost every  $\zeta \in \partial\Omega$ .

The proof follows at once from the fact that all boundary points  $\zeta \in \partial\Omega$  are  $(\phi, \alpha)$ -accessible, as pointed out above in Remark 3.4 (a).  $\square$

**Corollary 2B.** ([St98, Theorem 2, p. 307]) *Let  $f$  be a nonnegative subharmonic function on a domain  $G$  in  $\mathbb{R}^n$ ,  $G \neq \mathbb{R}^n$ ,  $n \geq 2$ , with  $C^1$  boundary. Let*

$$(5) \quad \int_G f(x)^p \delta(x)^\gamma dm(x) < \infty$$

for some  $p > 0$  and  $\gamma > -1 - \beta(p)$ . Let  $0 < d \leq n - 1$ . Then for each  $\tau \geq 1$  and  $\alpha > 0$  ( $\alpha > 1$  when  $\tau = 1$ ), there exists a subset  $E_\tau$  of  $\partial G$  with  $\mathcal{H}^d(E_\tau) = 0$  such that

$$\lim_{\rho \rightarrow 0} \left\{ \sup_{x \in \Gamma_{\tau, \alpha, \rho}(\zeta)} [\delta(x)^{n+\gamma-\frac{d}{\tau}} f(x)^p] \right\} = 0$$

for all  $\zeta \in \partial G \setminus E_\tau$ .

Above, for  $\zeta \in \partial G$  and  $\rho > 0$ ,

$$\Gamma_{\tau, \alpha, \rho}(\zeta) = \Gamma_{\tau, \alpha}(\zeta) \cap G_\rho,$$

where

$$\Gamma_{\tau, \alpha}(\zeta) = \{x \in G : |x - \zeta|^\tau < \alpha \delta(x)\}, \quad G_\rho = \{x \in G : \delta(x) < \rho\}.$$

Moreover,  $\beta(p) = \max\{(n-1)(1-p), 0\}$ .

Stoll makes the assumption  $\gamma > -1 - \beta(p)$  in order to exclude the trivial case  $f \equiv 0$ . As a matter of fact, it follows from a result of Suzuki [Su90, Theorem 2, p. 271] that (5) together with the condition  $\gamma \leq -1 - \beta(p)$  implies indeed that  $f \equiv 0$ , provided  $G$  is a bounded domain with  $C^2$  boundary. Unlike Stoll, we have imposed in Theorem 2 no restrictions on the exponent  $\gamma$  in order to exclude the trivial case  $u \equiv 0$ . Such possibilities are, however, referred in Remark 4.5 below.

#### 4. A LIMITING CASE RESULT TO NONINTEGRABILITY RESULTS OF SUZUKI

4.1. As another application of Theorem 1, we give in Corollary 3 below a supplement, or a limiting case result, to the following result of Suzuki.

**Suzuki's theorem.** ([Su91, Theorem and its proof, pp. 113–115]) *Let  $0 < p \leq 1$ . If a superharmonic (respectively nonnegative subharmonic) function  $v$  on  $\Omega$  satisfies*

$$\int_{\Omega} |v(x)|^p \delta(x)^{np-n-2p} dm(x) < \infty,$$

then  $v$  vanishes identically.

Suzuki pointed out that his result is sharp in the following sense: If  $p$ ,  $0 < p \leq 1$ , is fixed, then the exponent  $\gamma = np - n - 2p$  cannot be increased. On the other hand, clearly  $-n < \gamma \leq -2$ , when  $0 < p \leq 1$ . Since the class of permissible functions include, in addition to the functions  $t^p$ ,  $0 < p \leq 1$ , also a large amount of essentially different

functions, one is tempted to ask whether there exists any limiting case result for Suzuki's result, corresponding to the case  $p = 0$ . To be more precise, one may pose the following question:

*Let  $\Omega$  and  $v$  be as above. Let  $\gamma \leq -n$  and let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be permissible. Does the condition*

$$\int_{\Omega} \psi(|v(x)|) \delta(x)^\gamma dm(x) < \infty,$$

*imply  $v \equiv 0$ ?*

Observe that the least severe form of above integrability condition occurs when  $\gamma = -n$ .

**4.2.** Before giving an answer in Corollary 3, we state a general result for arbitrary  $\gamma \leq -2$ , which is, for  $-n < \gamma \leq -2$ , however, essentially more or less just Suzuki's theorem (see Remarks 4.3 (b) below). Our formulation has, however, the advantage that, unlike Suzuki's result, it contains a certain limiting case, Corollary 3, too.

**Theorem 3.** *Let  $\Omega$  be bounded. Let  $v$  be a superharmonic (respectively nonnegative subharmonic) function on  $\Omega$ . Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a strictly permissible function. Suppose*

$$(6) \quad \int_{\Omega} \psi(|v(x)|) \delta(x)^\gamma dm(x) < \infty,$$

*where  $\gamma \leq -2$  is such that there is a constant  $C = C(\gamma, n, \psi, \Omega) > 0$  for which*

$$(7) \quad s^{n+\gamma} \leq \psi(Cs^{n-2}) \text{ for all } s > \frac{1}{\text{diam } \Omega}.$$

*Then  $v$  vanishes identically.*

The proof is merely a slight modification of Suzuki's argument, combined with Theorem 1 above and also some additional estimates. For details, see [Ri03].  $\square$

**4.3. Remarks.** Next we consider the assumptions in Theorem 3.

- (a) The assumption  $\gamma \leq -2$  is unnecessary: If  $\gamma \in \mathbb{R}$ , then it follows easily from (7) and from the property (c) in 2.3 of strictly permissible functions that indeed  $\gamma \leq -2$ .
- (b) Suppose that  $-n < \gamma \leq -2$ . If, instead of (7), one supposes that

$$s^{n+\gamma} \leq \psi(Cs^{n-2}) \text{ for all } s > 0,$$

then clearly

$$\psi(|v(x)|) \geq C^{-\frac{n+\gamma}{n-2}} |v(x)|^{\frac{n+\gamma}{n-2}}$$

for all  $x \in \Omega$ . Thus (6) implies that

$$\int_{\Omega} |v(x)|^{\frac{n+\gamma}{n-2}} \delta(x)^\gamma dm(x) < \infty,$$

and hence  $v \equiv 0$  by Suzuki's theorem. Recall that here  $0 < p = \frac{n+\gamma}{n-2} \leq 1$  and  $\gamma = np - n - 2p$ . Thus Theorem 3, but now the assumption (7) replaced with the aforesaid assumption, is just a restatement of Suzuki's theorem for bounded domains.

- (c) If  $\gamma \leq -n$ , then the condition (7) clearly holds, since  $\psi$  is strictly permissible. This case gives indeed the already referred limiting case for Suzuki's result:

**Corollary 3.** *Let  $\Omega$  be bounded. Let  $v$  be a superharmonic (respectively nonnegative subharmonic) function on  $\Omega$ . Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be any strictly permissible function and let  $\gamma \leq -n$ . If*

$$\int_{\Omega} \psi(|v(x)|) \delta(x)^\gamma dm(x) < \infty,$$

*then  $v$  vanishes identically.*

For the proof observe that the condition (7) is indeed satisfied for  $\gamma \leq -n$ , since  $\Omega$  is bounded and  $\psi$  is increasing.  $\square$

**4.4. Remark.** The result of Theorem 3 does not, of course, hold any more, if one replaces strictly permissible functions by permissible functions. For a counterexample, set, say,  $v(x) = |x|^{2-n}$ ,  $\psi(t) = t^p$ , where  $\frac{n-1}{n-2} < p < \frac{n}{n-2}$ ,  $\gamma = np - n - 2p$  or just  $\gamma > 1$ . Then clearly

$$\int_B v(x)^p \delta(x)^\gamma dm(x) < \infty$$

but  $v \not\equiv 0$ .

**4.5. Remark.** Provided  $\Omega$  is bounded and  $\psi$  is strictly permissible, one can, with the aid of Theorem 3 and Corollary 3, exclude some trivial cases  $u \equiv 0$  from the result of Theorem 2 by imposing certain restrictions on the exponent  $\gamma$ . We point out only two cases:

- (i) By Corollary 3,  $\gamma > -n$ , regardless of  $\psi$ .
- (ii) By Suzuki's theorem,  $\gamma > np - n - 2p$ , in the case when  $\psi(t) = t^p$ ,  $0 < p \leq 1$ .

## 5. EXCEPTIONAL SETS FOR SUBHARMONIC, PLURISUBHARMONIC AND CONVEX FUNCTIONS

**5.1. Previous results.** Blanchet [Bl95, Theorems 3.1, 3.2 and 3.3, pp. 312–313] gave the following removability results.

**Blanchet's theorem.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $S$  be a hypersurface of class  $\mathcal{C}^1$  which divides  $\Omega$  into two subdomains  $\Omega_1$  and  $\Omega_2$ . Let  $u \in \mathcal{C}^0(\Omega) \cap \mathcal{C}^2(\Omega_1 \cup \Omega_2)$  be subharmonic (respectively convex (or respectively plurisubharmonic provided  $\Omega$  is then a domain in  $\mathbb{C}^n$ ,  $n \geq 1$ )). If  $u_i = u|_{\Omega_i} \in \mathcal{C}^1(\Omega_i \cup S)$ ,  $i = 1, 2$ , and*

$$(8) \quad \frac{\partial u_i}{\partial \bar{n}^k} \geq \frac{\partial u_k}{\partial \bar{n}^k}$$

on  $S$  with  $i, k = 1, 2$ , then  $u$  is subharmonic (respectively convex (or respectively plurisubharmonic)) in  $\Omega$ .

Above  $\bar{n}^k = (\bar{n}_1^k, \dots, \bar{n}_n^k)$  is the unit normal exterior to  $\Omega_k$ , and  $u_k \in \mathcal{C}^1(\Omega_k \cup S)$ ,  $k = 1, 2$ , means that there exist  $n$  functions  $v_k^j$ ,  $j = 1, \dots, n$ , continuous on  $\Omega_k \cup S$ , such that

$$v_k^j(x) = \frac{\partial u_k}{\partial x_j}(x)$$

for all  $x \in \Omega_k$ ,  $k = 1, 2$  and  $j = 1, \dots, n$ .

Instead of hypersurfaces of class  $\mathcal{C}^1$ , we will below allow arbitrary sets of finite  $(n-1)$ -dimensional (respectively  $(2n-1)$ -dimensional) Hausdorff measure as exceptional sets. Then we must, however, replace the condition (8) by another, related condition, the condition (iv) in Theorem 4 below. In the case of subharmonic and plurisubharmonic functions, we must also impose an additional integrability condition on the second partial derivatives  $\frac{\partial^2 u}{\partial x_j^2}$ ,  $j = 1, \dots, n$ . Observe that in the case of (separately) convex functions we do not, unlike Blanchet, need any smoothness assumptions of the functions (except continuity). Our method of proof is rather elementary, thus natural, with the only exception that we need one geometric measure theory result of Federer.

**5.2. The case of subharmonic functions.** The following measure theoretic result is essential:

**Lemma 1.** ([Fe69, Theorem 2.10.25, p. 188]) *Suppose that  $A \subset \mathbb{R}^n$ ,  $n \geq 2$ , is such that  $\mathcal{H}^{n-1}(A) < \infty$ . Then for all  $j$ ,  $1 \leq j \leq n$ , and for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$  the set  $A(X_j)$  is finite.*

Our result is:

**Theorem 4.** *Suppose that  $\Omega$  is a domain in  $\mathbb{R}^n$  (respectively in  $\mathbb{C}^n$ ),  $n \geq 2$ . Let  $E \subset \Omega$  be closed in  $\Omega$  and  $\mathcal{H}^{n-1}(E) < \infty$  (respectively  $\mathcal{H}^{2n-1}(E) < \infty$ ). Let  $u : \Omega \rightarrow \mathbb{R}$  be such that*

- (i)  $u \in \mathcal{C}^0(\Omega)$ ,
- (ii)  $u \in \mathcal{C}^2(\Omega \setminus E)$ ,
- (iii) for each  $j$ ,  $1 \leq j \leq n$  (respectively  $1 \leq j \leq 2n$ ),  $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}_{\text{loc}}^1(\Omega)$ ,

(iv) for each  $j$ ,  $1 \leq j \leq n$  (respectively  $1 \leq j \leq 2n$ ), and for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$  (respectively for  $\mathcal{H}^{2n-1}$ -almost all  $X_j \in \mathbb{R}^{2n-1}$ ) such that  $E(X_j)$  is finite, one has

$$\liminf_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \leq \limsup_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \varepsilon, X_j)$$

for each  $x_j^0 \in E(X_j)$ ,

(v)  $u$  is subharmonic (respectively separately subharmonic) in  $\Omega \setminus E$ .

Then  $u$  is subharmonic (respectively separately subharmonic).

*Proof.* We consider only the subharmonic case. It is sufficient to show that

$$\int u(x) \Delta \varphi(x) dx \geq 0$$

for all nonnegative testfunctions  $\varphi \in \mathcal{D}(\Omega)$ . Since  $u \in C^2(\Omega \setminus E)$  and  $u$  is subharmonic in  $\Omega \setminus E$ ,  $\Delta u(x) \geq 0$  for all  $x \in \Omega \setminus E$ . Therefore the claim follows if we show that

$$\int u(x) \Delta \varphi(x) dx \geq \int \Delta u(x) \varphi(x) dx.$$

For this purpose fix  $j$ ,  $1 \leq j \leq n$ , for a while. By Fubini's theorem,

$$\int u(x) \frac{\partial^2 \varphi}{\partial x_j^2}(x) dx = \int \left[ \int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j \right] dX_j.$$

Using Lemma 1, assumptions (iii), (iv) and Fubini's theorem, we see that for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$ ,

$$(9) \quad \begin{cases} E(X_j) \text{ is finite,} \\ \frac{\partial^2 u}{\partial x_j^2}(\cdot, X_j) \in \mathcal{L}_{\text{loc}}^1(\Omega(X_j)), \\ \liminf_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \leq \limsup_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \varepsilon, X_j) \text{ for all } x_j^0 \in E(X_j). \end{cases}$$

Let  $K = \text{spt} \varphi$ . Choose a domain  $\Omega_1$  such that  $K \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$  and  $\overline{\Omega_1}$  is compact. Since  $E(X_j)$  is finite, there is  $M = M(X_j) \in \mathbb{N}$  such that  $E(X_j) = \{x_j^1, \dots, x_j^M\}$  where  $x_j^k < x_j^{k+1}$ ,  $k = 1, \dots, M-1$ . Choose for each  $k = 1, \dots, M$  real numbers  $a_k, b_k \in (\Omega \setminus E)(X_j)$  such that  $a_k < x_j^k < b_k = a_{k+1} < x_j^{k+1} < b_{k+1}$ ,  $k = 1, \dots, M-1$ , and  $a_1, b_M \in (\Omega_1 \setminus (E \cup K))(X_j)$ . Then

$$(10) \quad \int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j = \sum_{k=1}^M \int_{a_k}^{b_k} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j.$$

Fix  $k$ ,  $1 \leq k \leq M$ , arbitrarily, and write  $a = a_k$ ,  $b = b_k$ ,  $x_j^0 = x_j^k$ . Then

$$\begin{aligned}
\int_a^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j &= \int_a^{x_j^0} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j + \int_{x_j^0}^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j \\
&= \lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j + \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j \\
&= \lim_{\varepsilon \rightarrow 0+0} \left[ \int_a^{x_j^0 - \varepsilon} u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j - \int_a^{x_j^0 - \varepsilon} \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j \right] + \\
&\quad + \lim_{\varepsilon \rightarrow 0+0} \left[ \int_{x_j^0 + \varepsilon}^b u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j - \int_{x_j^0 + \varepsilon}^b \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j \right] \\
&= \lim_{\varepsilon \rightarrow 0+0} \left[ \int_a^{x_j^0 - \varepsilon} u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j - \lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j \right] + \\
&\quad + \lim_{\varepsilon \rightarrow 0+0} \left[ \int_{x_j^0 + \varepsilon}^b u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j - \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) dx_j \right] \\
&= u(b, X_j) \frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \varphi}{\partial x_j}(a, X_j) + \\
&\quad - \lim_{\varepsilon \rightarrow 0+0} \left[ \int_a^{x_j^0 - \varepsilon} \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j - \int_a^{x_j^0 - \varepsilon} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j \right] + \\
&\quad - \lim_{\varepsilon \rightarrow 0+0} \left[ \int_{x_j^0 + \varepsilon}^b \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j - \int_{x_j^0 + \varepsilon}^b \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j \right].
\end{aligned}$$

Since  $\frac{\partial^2 u}{\partial x_j^2}(\cdot, X_j) \in \mathcal{L}_{\text{loc}}^1(\Omega(X_j))$ , the limits

$$\lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j$$

exist. Thus also the limits

$$\lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j$$

exist. Therefore, remembering also that  $a, b \in (\Omega \setminus E)(X_j)$ , we get

$$(11) \quad \lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j = \lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \varphi(x_j^0, X_j) - \frac{\partial u}{\partial x_j}(a, X_j) \varphi(a, X_j)$$

and

$$(12) \quad \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b \frac{\partial u}{\partial x_j}(x_j, X_j) \varphi(x_j, X_j) dx_j = \frac{\partial u}{\partial x_j}(b, X_j) \varphi(b, X_j) - \lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \varepsilon, X_j) \varphi(x_j^0, X_j).$$

(The limits

$$\lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \varepsilon, X_j)$$

indeed exist for all points  $x_j^0 \in (\Omega \setminus E)(X_j)$ , for which  $\varphi(x_j^0, X_j) > 0$ .)

Using (11) and (12) and also the assumption (iv), we get

$$\begin{aligned}
\int_a^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j &= [u(b, X_j) \varphi(b, X_j) - u(a, X_j) \varphi(a, X_j)] + \\
&+ [-\frac{\partial u}{\partial x_j}(b, X_j) \varphi(b, X_j) + \frac{\partial u}{\partial x_j}(a, X_j) \varphi(a, X_j)] + [\lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \varepsilon, X_j) - \lim_{\varepsilon \rightarrow 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \varepsilon, X_j)] \varphi(x_j^0, X_j) + \\
&+ \lim_{\varepsilon \rightarrow 0+0} \int_a^{x_j^0 - \varepsilon} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j + \lim_{\varepsilon \rightarrow 0+0} \int_{x_j^0 + \varepsilon}^b \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j \geq \\
&\geq [u(b, X_j) \varphi(b, X_j) - u(a, X_j) \varphi(a, X_j)] + [-\frac{\partial u}{\partial x_j}(b, X_j) \varphi(b, X_j) + \frac{\partial u}{\partial x_j}(a, X_j) \varphi(a, X_j)] + \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j.
\end{aligned}$$

In view of this and of (10) we get

$$\int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j \geq \int \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) dx_j.$$

Integrating then here on both sides with respect to  $X_j \in \mathbb{R}^{N-1}$ , and using (9) and also Fubini's theorem, we get

$$\int u(x) \frac{\partial^2 \varphi}{\partial x_j^2}(x) dx \geq \int \frac{\partial^2 u}{\partial x_j^2}(x) \varphi(x) dx.$$

Hence

$$\int u(x) \Delta \varphi(x) dx_j \geq \int \Delta u(x) \varphi(x) dx \geq 0,$$

concluding the proof.  $\square$

**Corollary 4A.** Suppose that  $\Omega$  is a domain in  $\mathbb{R}^n$  (respectively in  $\mathbb{C}^n$ ),  $n \geq 2$ . Let  $E \subset \Omega$  be closed in  $\Omega$  and  $\mathcal{H}^{n-1}(E) < \infty$  (respectively  $\mathcal{H}^{2n-1}(E) < \infty$ ). Let  $u : \Omega \rightarrow \mathbb{R}$  be such that

- (i)  $u \in \mathcal{C}^1(\Omega)$ ,
- (ii)  $u \in \mathcal{C}^2(\Omega \setminus E)$ ,
- (iii) for each  $j$ ,  $1 \leq j \leq n$  (respectively  $1 \leq j \leq 2n$ ),  $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}_{\text{loc}}^1(\Omega)$ ,
- (iv)  $u$  is subharmonic (respectively separately subharmonic) in  $\Omega \setminus E$ .

Then  $u$  is subharmonic (respectively separately subharmonic).

**5.3. The case of plurisubharmonic functions.** In order to obtain a similar result for plurisubharmonic functions, we need the following result of Lelong.

**Lemma 2.** ([Le69, Theorem 1, p. 18]) Suppose that  $D$  is a domain of  $\mathbb{C}^n$ ,  $n \geq 2$ . Let  $v : D \rightarrow [-\infty, +\infty)$ . Then  $v$  is plurisubharmonic if and only if the following condition holds:

For each  $z_0 \in D$  and for each affine transformation  $A = (A_1, \dots, A_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,

$$\begin{aligned}
z' = Az &\Leftrightarrow (z'_1, \dots, z'_n) = (A_1(z_1, \dots, z_n), \dots, A_n(z_1, \dots, z_n)) \\
&\Leftrightarrow \begin{cases} z'_1 = A_1(z_1, \dots, z_n) = z_1^0 + a_{11}z_1 + \dots + a_{1n}z_n, \\ \vdots \\ z'_n = A_n(z_1, \dots, z_n) = z_n^0 + a_{n1}z_1 + \dots + a_{nn}z_n, \end{cases}
\end{aligned}$$

for which  $\det A \neq 0$ , the function  $v \circ A : A^{-1}(D) \rightarrow [-\infty, +\infty)$  is subharmonic.

**Corollary 4B.** Suppose that  $\Omega$  is a domain of  $\mathbb{C}^n$ ,  $n \geq 2$ . Let  $E \subset \Omega$  be closed in  $\Omega$  and  $\mathcal{H}^{2n-1}(E) < \infty$ . Let  $u : \Omega \rightarrow \mathbb{R}$  be such that

- (i)  $u \in \mathcal{C}^1(\Omega)$ ,
- (ii)  $u \in \mathcal{C}^2(\Omega \setminus E)$ ,
- (iii) for each  $j$ ,  $1 \leq j \leq 2n$ ,  $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}_{\text{loc}}^1(\Omega)$ ,
- (iv)  $u$  is plurisubharmonic in  $\Omega \setminus E$ .

Then  $u$  is plurisubharmonic.

*Proof.* By Lemma 2 it is sufficient to show that  $v = u \circ A$  is subharmonic in  $\Omega' = A^{-1}(\Omega)$  for any affine mapping  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$  with  $\det A \neq 0$ . Clearly  $v \in \mathcal{C}^1(\Omega')$  and  $v \in \mathcal{C}^2(\Omega' \setminus E')$ , where  $E' = A^{-1}(E)$ . It is easy to see that for each  $j$ ,  $1 \leq j \leq 2n$ ,  $\frac{\partial^2 v}{\partial x_j^2} \in \mathcal{L}_{\text{loc}}^1(\Omega')$ . Since  $u$  is plurisubharmonic in  $\Omega \setminus E$ ,  $v$  is by Lemma 2 subharmonic in  $\Omega' \setminus E'$ , thus subharmonic in  $\Omega'$  by Corollary 4A.  $\square$

#### 5.4. The case of convex functions.

We recall first some very basic properties of convex functions. Let  $D$  be a domain of  $\mathbb{R}^n$ ,  $n \geq 1$ . A function  $f : D \rightarrow \mathbb{R}$  is *convex* if the following condition is satisfied: For each  $x, y \in D$  such that  $\{tx + (1-t)y : t \in [0, 1]\} \subset D$ , one has  $f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$  for all  $t \in [0, 1]$ .

**Lemma 3.** ([We94, Theorem 5.1.3, p. 195]) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Then  $f$  possesses left and right derivatives at each interior point of  $[a, b]$ , and if  $x_1, x_2$  are interior points of  $[a, b]$  with  $x_1 < x_2$ , then*

$$-\infty < f'_-(x_1) \leq f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2) \leq f'_+(x_2) < +\infty.$$

**Lemma 4.** ([We94, Theorem 5.1.8, p. 198]) *Let  $f : (a, b) \rightarrow \mathbb{R}$ . Then  $f$  is convex if and only if it has support at each point of  $(a, b)$ , i.e. for any  $x_0 \in (a, b)$  there is a constant  $m \in \mathbb{R}$  such that*

$$f(x_0) + m(x - x_0) \leq f(x)$$

for all  $x \in (a, b)$ .

Moreover, if  $f$  is convex, then any  $m$ ,  $f'_-(x_0) \leq m \leq f'_+(x_0)$ , will do.

We consider first separately convex functions:

**Theorem 5.** *Suppose that  $\Omega$  is a domain of  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $E \subset \Omega$  be closed in  $\Omega$  and  $\mathcal{H}^{n-1}(E) < \infty$ . Let  $u : \Omega \rightarrow \mathbb{R}$  be such that*

(i)  $u \in \mathcal{C}^0(\Omega)$ ,

(ii) *for each  $j$ ,  $1 \leq j \leq n$ , and for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$  such that  $E(X_j)$  is finite, one has*

$$\liminf_{\varepsilon \rightarrow 0+0} \frac{\partial_{-} u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \leq \limsup_{\varepsilon \rightarrow 0+0} \frac{\partial_{+} u}{\partial x_j}(x_j^0 + \varepsilon, X_j)$$

*for each  $x_j^0 \in E(X_j)$ ,*

(iii)  *$u$  is separately convex in  $\Omega \setminus E$ .*

Then  $u$  is separately convex.

Above, and in the sequel,  $\frac{\partial_{-} u}{\partial x_j}(x_j, X_j)$  and  $\frac{\partial_{+} u}{\partial x_j}(x_j, X_j)$ ,  $j = 1, \dots, n$ , are the left and right partial derivatives of  $u$ , respectively, taken at the point  $x = (x_j, X_j)$ .

Observe that the condition (ii) is a necessary condition for (separately) convex functions.

*Proof of Theorem 5.* Choose  $j$ ,  $1 \leq j \leq n$ , arbitrarily. Using Lemma 1 and the condition (ii) we see that for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$ ,

$$\begin{cases} E(X_j) \text{ is finite,} \\ (\Omega \setminus E)(X_j) \ni x_j \mapsto u(x_j, X_j) \in \mathbb{R} \text{ is convex,} \\ \liminf_{\varepsilon \rightarrow 0+0} \frac{\partial_{-} u}{\partial x_j}(x_j^0 - \varepsilon, X_j) \leq \limsup_{\varepsilon \rightarrow 0+0} \frac{\partial_{+} u}{\partial x_j}(x_j^0 + \varepsilon, X_j) \text{ for all } x_j^0 \in E(X_j). \end{cases}$$

Using this, Lemma 3 and Lemma 4 one sees that for  $\mathcal{H}^{n-1}$ -almost all  $X_j \in \mathbb{R}^{n-1}$  the functions

$$(13) \quad \Omega(X_j) \ni x_j \mapsto u(x_j, X_j) \in \mathbb{R}$$

are in fact convex. (Here one proceeds e.g. as follows: Suppose that  $(a, b)$  is an arbitrary interval of the open set  $\Omega(X_j)$ , that  $E(X_j) \cap (a, b) = \{x_j^1, \dots, x_j^N\}$ , where  $a < x_j^1 < x_j^2 < \dots < x_j^{N-1} < x_j^N = b$ ,  $k = 1, \dots, N-1$  and  $x_j^{N+1} = b$ . If  $u(\cdot, X_j)|(a, x_j^k)$  and  $u(\cdot, X_j)|(x_j^k, x_j^{k+1})$ , are convex, then it follows from the assumptions that  $u(\cdot, X_j)|(a, x_j^{k+1})$ , is convex,  $k =$

$1, \dots, N$ .) From this and from the fact that  $u$  is continuous, it follows easily that the functions of the form (13) above are in fact convex for all  $X_j \in \mathbb{R}^{n-1}$ . Since  $j$ ,  $1 \leq j \leq n$ , was arbitrary, the claim follows.  $\square$

**Corollary 5.** Suppose that  $\Omega$  is a domain of  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $E \subset \Omega$  be closed in  $\Omega$  and  $\mathcal{H}^{n-1}(E) < \infty$ . Let  $u : \Omega \rightarrow \mathbb{R}$  be such that

- (i)  $u \in C^1(\Omega)$ ,
- (ii)  $u$  is (separately) convex in  $\Omega \setminus E$ .

Then  $u$  is (separately) convex.

The separately convex case follows directly from Theorem 5. The convex case follows from the separately convex case with the aid of the following Lelong type result (whose proof is similar to [Le69, proof of Theorem 1, p. 18]).  $\square$

**Lemma 5.** Suppose that  $D$  is a domain of  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $v : D \rightarrow [-\infty, +\infty)$ . Then  $v$  is convex if and only if the following condition holds:

For each  $x_0 \in D$  and for each affine transformation  $A = (A_1, \dots, A_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,

$$\begin{aligned} x' = Ax &\Leftrightarrow (x'_1, \dots, x'_n) = (A_1(x_1, \dots, x_n), \dots, A_n(x_1, \dots, x_n)) \\ &\Leftrightarrow \begin{cases} x'_1 = A_1(x_1, \dots, x_n) = x_1^0 + a_{11}x_1 + \dots + a_{1n}x_n, \\ \vdots \\ x'_n = A_n(x_1, \dots, x_n) = x_n^0 + a_{n1}x_1 + \dots + a_{nn}x_n, \end{cases} \end{aligned}$$

for which  $\det A \neq 0$ , the function  $v \circ A : A^{-1}(D) \rightarrow [-\infty, +\infty)$  is separately convex.

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